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Classical electrodynamics and the definition of an energy tensor for a system of charged particles and electromagnetic fields

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Abstract. Using an invariant spatial volume element the energy-density tensor $T^{\mu\nu}$ is integrated over a hyperplane orthogonal to the velocity V^{λ} of the observer. The resulting energy tensor $\mathscr{T}^{\mu\nu}$ for the system yields the momentum and energy of the system relative to a given observer in the usual way. It is shown that the usual conservation theorems for the momentum of a free field and the momentum radiated by an accelerated particle are recovered, but the expression obtained for the (bound) velocity-field momentum differs from the usual expression of this quantity given in the literature. The new definition of momentum results in a rational definition when applied to two or more particles, which is in contradistinction to the usual definition which cannot be generalised for two or more particles unambiguously. The definition of $\mathscr{T}^{\mu\nu}$ given here is the flat space–time specialisation of a definition previously given by the author in the context of general relativity. Thus a uniform prescription for the treatment of problems concerning energy and momentum is achieved together with the resolution of a long-standing conceptual problem.

1. Introduction

The tensor $T^{\mu\nu}$ is known by a variety of names, e.g. energy–momentum--stress tensor, stress–energy tensor, energy–momentum tensor, energy tensor. The components of $T^{\mu\nu}$ are related to the density of momentum and the density of energy by contracting once and twice respectively with the unit vector tangent to the observer's world line, V^{λ}. Thus $T^{\mu\nu}V_{\nu}$ gives the momentum density, and $T^{\mu\nu}V_{\nu}V_{\mu}$ gives the energy density. For a more detailed account of the interpretation of $T^{\mu\nu}$ see Misner *et al* (1973) p 131. For readers unaquainted with the geometrical description of physical quantities as invariants, and for whom some of the expressions given here might appear in an unfamiliar form, I have included a brief discussion of the geometrical formulation in an appendix.

In order to obtain the energy, the energy density must be integrated over the appropriate spatial volume, and this must be done in an invariant way. Similarly the momentum is the momentum density integrated covariantly. This can be achieved by defining a spatial volume integral of the tensor $T^{\mu\nu}$ in a covariant way resulting in a new tensor $\mathscr{T}^{\mu\nu}$. The momentum and energy are obtained directly from $\mathscr{T}^{\mu\nu}$ by contraction with V^{λ}. A natural name for $\mathscr{T}^{\mu\nu}$ is the energy tensor (or energy–momentum tensor) of the system over which the integration is taken. I shall call $T^{\mu\nu}$ the energy-density tensor in order to distinguish it from the energy tensor $\mathscr{T}^{\mu\nu}$.

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The definition of $\mathcal{T}^{\mu\nu}$ which I give is the flat space–time specialisation of a definition for an energy tensor that I have previously given in general relativity (Oliver 1977). In general relativity it is manifest that the energy-density tensor has a central role in the theory and thus forms a natural starting point for a definition of energy and momentum. Classical electrodynamics can be formulated so that the energy-density tensor is given a central role in which case the energy and momentum are defined quite naturally in terms of it. This way of formulating the theory has the philosophical advantage of aiding the methodological and structural unification of these two theories.

In order to make the advantages of this approach clear I first describe the usual procedure. Conventionally the energy-density tensor $T^{\mu\nu}$ is contracted with an infinitesimal hyperplane element $d\sigma^{\nu}$. $d\sigma^{\nu} = n^{\nu}\|d\sigma\|$, where n^{ν} is the unit time-like vector normal to the hyperplane σ and pointing into the future, so $\|d\sigma\| = -n_{\nu} . d\sigma^{\nu}$. The resulting infinitesimal vector $T^{\mu\nu}\, d\sigma_{\nu}$ is integrated over the hyperplane σ to give

$$P^{\mu} = \int_{\sigma} T^{\mu\nu}\, d\sigma_{\nu}. \tag{1}$$

P^{μ} is taken for the momentum of the system, with the choice of σ depending on the physical system to which the definition is applied. It is this uncertainty in the specification of σ which has led to ambiguities in the treatment of the momentum of many-particle systems.

One subtle problem is how the measure of spatial volume should be defined. In (1) the measure $d\sigma_{\nu}$ depends on the choice of σ, but σ is not specified as part of the definition; the hyperplane σ is chosen differently in different applications. Whereas in the definition I give the hyperplane of integration is specified as part of the definition and is the same for every application. The measure of spatial volume which I use can be described as the magnitude of that part of $d\sigma_{\nu}$ orthogonal to the observer's velocity. In order to show that this is indeed a rational choice I give a discussion of its definition and show how it is related to a corresponding definition of length; this I do in § 2. In § 3 the definition of $\mathcal{T}^{\mu\nu}$ is given.

In § 4 and § 5 I apply the formalism to the problems of the momentum of a free field and the radiation momentum from a particle, and show that the usual well known results follow. In § 6 the formalism is applied to the problem of the velocity-field momentum and the resulting expressions differ from the conventional ones to which they correspond; reasons for believing that this does not provide an argument against the proposed definition of $\mathcal{T}^{\mu\nu}$ are given. When applied to many-particle systems the definition of $\mathcal{T}^{\mu\nu}$ results in an explicit and conceptually simple expression, whereas the conventional approach results in confusion.

Notation. The vectors and tensors employed in this paper all reside on the four-dimensional manifold of space–time. The n-vectors ($n \leq 4$) formed by the wedge product $\wedge$ of n vectors are completely antisymmetric tensors of order n, thus a one-vector is just a vector. Vectors and tensors are usually denoted by a typical component, e.g. v^{λ}, $T^{\mu\nu}$, but I do use a coordinate-free notation on occasion, e.g. v, T. The dot product is defined by $v . T = v^{\lambda} g_{\lambda\mu} T^{\mu\nu}$ and the norm by $\|v\| = |v . v|^{1/2}$. The dual of a vector, or tensor, is obtained by contraction with the Levi–Civita completely antisymmetric tensor. The dual is denoted by an asterisk; thus $^{*}F$ is the dual of F. The signature of $g_{\mu\nu}$ is +2. In matters concerning electrodynamics my conventions are the same as Rohrlich (1965) with the exception that I have put $c = 1$ throughout and use τ for the observer's proper time and s_k for the proper time of the kth particle. Where no confusion arises I have usually omitted the subscript on s, e.g. $v_k(s_k) \equiv v_k(s)$. In the expressions involving integrals the coordinate systems are assumed to be rectilinear. This restriction could be dropped at the expense of complicating the mathematical expression of the integrals (see § 3).

2. Invariant spatial volume

There are two simple invariant measures of the spatial length of a rod:

(1) what might be called its measure of intrinsic length which is independent of the observer, due to Lorentz (1923);

(2) the observer-dependent measure of its length.

In figure 1, A and A′ are arbitrary events on the world lines L and L′ of the ends of the rod. B is the event on L′ such that $\overrightarrow{BA}$ is orthogonal to the vector tangent to L′ at B. A and P are the events where the world lines L and L′ intersect the hyperplane Σ which is orthogonal to the world line of the observer and intersects it at the event X at the observer's proper time τ. I sometimes exhibit τ explicitly by writing Σ as $\Sigma(\tau)$.

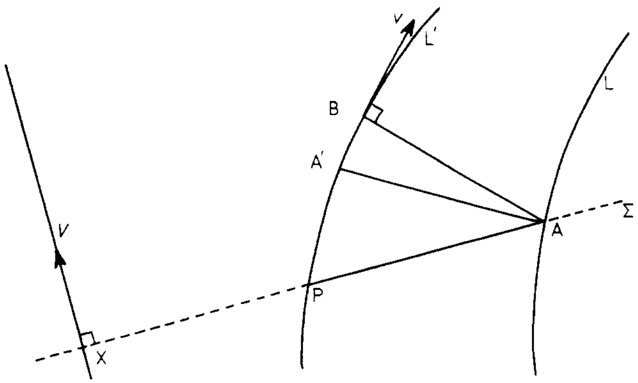

Figure 1.

Write the vector $\overrightarrow{AA'} = dx$; let v and V be the unit vectors tangent to L′ at B and L at X, respectively. From figure 1 it is clear that

$$dl(v) \equiv \overrightarrow{AB} = (I + vv) \,.\, dx$$

$$dl(V) \equiv \overrightarrow{AP} = (I + VV) \,.\, dx$$

where I is the unit tensor. The lengths are the norms of these vectors, thus:

(1) The invariant measure of intrinsic length is defined to be

$$\|dl(v)\| = |dx \,.\, dx + (v \,.\, dx)^2|^{1/2}.$$

(2) The observer-dependent invariant measure of length is defined to be

$$\|dl(V)\| = |dx \,.\, dx + (V \,.\, dx)^2|^{1/2}.$$

It is easy to see that the intrinsic length measure exhibits the Fitzgerald–Lorentz contraction whereas the observer-dependent measure does not.

Consider a small parallelepiped shaped body with edges defined by the vectors $\overrightarrow{AA'}$, $\overrightarrow{AA''}$, $\overrightarrow{AA'''}$ where A, A', A'', A''' are events on the world lines L, L', L'', L''' of four adjacent vertices. Events C, Q on L'', D, R on L''' corresponding to B, P on L' in figure 1 enable vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ and $\overrightarrow{AP}$, $\overrightarrow{AQ}$, $\overrightarrow{AR}$ to be defined. The former set allow the construction of the intrinsic hyperarea, i.e. the intrinsic spatial volume, of the body analogously to the intrinsic spatial length. The latter set of vectors, which all lie in the hyperplane Σ, lead to the observer-dependent measure of spatial volume. The algebra is straightforward (Oliver 1977) and on writing $\overrightarrow{AA'} = dx'$, $\overrightarrow{AA''} = dx''$, $\overrightarrow{AA'''} = dx'''$ gives for the intrinsic spatial volume

$$dV(v) = \pm\sqrt{(-g)}\det\begin{pmatrix} v^1 & dx'^1 & dx''^1 & dx'''^1 \\ v^2 & dx'^2 & dx''^2 & dx'''^2 \\ v^3 & dx'^3 & dx''^3 & dx'''^3 \\ v^4 & dx'^4 & dx''^4 & dx'''^4 \end{pmatrix} = \pm\sqrt{(-g)}v \wedge dx' \wedge dx'' \wedge dx'''.$$

The factor $\sqrt{(-g)}$ is necessary in order that $dV(v)$ is an invariant, and the sign is chosen to make $dV(v)$ positive. The observer-dependent spatial volume is given by the same expression on replacing v by V, namely $dV(V)$.

Synge and Schild (1949, § 7.3) make the important distinction between the metrical idea of volume and the more fundamental non-metrical concept of extension. The three-extension of the region of spacetime delimited by the three vectors dx', dx'', dx''' is defined to be the three-vector $d\tau_{(3)} = dx' \wedge dx'' \wedge dx'''$. The dual of this three-vector is the vector hyperplane element $d\sigma \equiv {}^*d\tau_{(3)}$; in components this is $d\sigma_\nu = \epsilon_{\nu\iota\lambda\mu}\, dx'^\iota\, dx''^\lambda\, dx'''^\mu$, where $\epsilon_{\alpha\beta\gamma\delta}$ is the completely antisymmetric tensor of Levi-Civita. Synge and Schild define the three-volume corresponding to the three-extension as the norm of the three-extension, $dv_{(3)} = \|d\tau_{(3)}\| \equiv \|d\sigma\|$.

Let n be the unit vector normal to the hyperplane containing dx', dx'', dx'''. Since the orientation of $d\tau_{(3)}$ is normal to this hyperplane we have, from the definition of $dv_{(3)}$,

$$dv_{(3)} = \pm\sqrt{(-g)}n \wedge d\tau_{(3)} \equiv dV(n)$$

using the definition of $dV(\cdot)$. Also since $d\sigma$ is normal to the hyperplane we have $d\sigma = n\, dv_{(3)} \equiv d\sigma(n)$, where the argument of $d\sigma(n)$ indicates the orientation of the hyperplane element. Hence we have the useful result that

$$n\, dV(n) = d\sigma(n). \tag{2}$$

In the literature either $d\sigma$ or $\|d\sigma\|$ is used as the measure of spatial volume, as for example in (1). In order to see that this is, in general, a mistake consider the following correspondences. First remark that the vector dx and its norm $\|dx\|$ can be written as the one-vector and one-volume (i.e. length) $d\tau_{(1)}$ and $dv_{(1)}$ respectively, and notice that $\|dx\|$ is a space–time interval. Now consider the geometrical meaning of the three quantities dx, ds, $\|dl\|$, i.e. $d\tau_{(1)}$, $dv_{(1)}$, $\|dl\|$, and compare them with the three quantities $d\tau_{(3)}$, $dv_{(3)}$, dV. Neither the vector dx ($\equiv d\tau_{(1)}$) nor the space–time interval $ds \equiv \|dx\|$ ($\equiv dv_{(1)}$) can be given the role of the measure of spatial distance (length), whereas the vector dl and its norm $\|dl\|$ has just this significance. Therefore the correct measure of spatial volume must be the corresponding quantity defined in a three-dimensional hypersurface, namely dV. Manifestly $d\tau_{(3)}$ and $dv_{(3)}$ cannot in general be associated with the notion of spatial volume. It is the norms of the projections of $d\tau_{(3)}$ and $d\tau_{(1)}$ onto a specified hyperplane which are identified as the spatial volume and spatial length.

Synge (1965, ch I, § 16) gives an account of the intrinsic length, and in ch VIII, § 6 an account of cross sections of world tubes, i.e. spatial volumes, but he does not demonstrate their relation. My $\mathrm{d}V(v)$ is what Synge calls the normal cross section of the world tube, and $\mathrm{d}V(V)$ is the cross section with normal V.

The purpose of the argument of this section is to draw attention to the intimate relationship between $\|\mathrm{d}l(V)\|$ and $\mathrm{d}V(V)$. Also to point out the misuse of $\mathrm{d}v_{(3)}$ as a measure of spatial volume.

3. The energy tensor, momentum vector and energy invariant

In order to obtain the energy tensor for an infinitesimal spatial region of space–time from the energy-density tensor the latter must be multiplied by the invariant measure of spatial volume corresponding to the region. No particles are necessarily present in the region so that the natural measure of the spatial volume of the region is $\mathrm{d}V(V)$. Thus we obtain

$$\mathrm{d}\mathcal{T}^{\mu\nu} = T^{\mu\nu}\,\mathrm{d}V(V)$$

for the energy tensor for the infinitesimal region.

Since it is understood that rectilinear coordinates are used we can integrate to get the energy tensor

$$\mathcal{T}^{\mu\nu}(\tau) = \int_{\Sigma(\tau)} T^{\mu\nu}\,\mathrm{d}V(V). \qquad (3)$$

V and $\Sigma(\tau)$ are defined as in § 2. N.B. If curvilinear coordinates are to be used then shifters must be employed in order to allow for the change in the component values of T as it is moved to a common space–time event before integration, see Toupin (1956, § 3) and Ericksen (appendix to Truesdell and Toupin 1960, §§ 16, 17).

The momentum and energy are defined in terms of $\mathcal{T}^{\mu\nu}$ by

$$\mathcal{P}^{\mu} = \mathcal{T}^{\mu\nu}V_{\nu}, \qquad \mathcal{E} = \mathcal{T}^{\mu\nu}V_{\nu}V_{\mu}. \qquad (4)$$

To see that this is very reasonable recall that $T^{\mu\nu}V_{\nu}$ and $T^{\mu\nu}V_{\nu}V_{\mu}$ are interpreted as the density of momentum and the density of energy (or mass–energy if you prefer), for details of this interpretation see Misner *et al* (1973, p 131). To obtain the momentum and energy these quantities must be integrated over the relevant spatial volume. Now, remark that the observer's velocity V is independent of the integration and may be taken out from under the integral sign, thus leading to the expressions for $\mathcal{P}$ and $\mathcal{E}$ given in equation (4).

A simple example will illustrate the procedure. I construct the energy-density tensor for a particle for the phenomenon of intertia; from this will come the intertial momentum and kinetic energy. Recall the definition of the charge current density

$$j^{\mu}(x) = e\int_{-\infty}^{\infty} \delta(x - z(s))v^{\mu}(s)\,\mathrm{d}s$$

which is manifestly covariant. Define $T^{\mu\nu}$ in an analogous way as

$$T^{\mu\nu}(x) = m\int_{-\infty}^{\infty} \delta(x - z(s))v^{\mu}(s)v^{\nu}(s)\,\mathrm{d}s \qquad (5)$$

where s is the proper time of the particle. This is the accepted definition.

With the energy-density tensor given in (5) and using the identity $v\,.\,V\,\mathrm{d}s\,\mathrm{d}V(V) \equiv \mathrm{d}^4x$, proved below, we have from (3)

$$\mathscr{T}^{\mu\nu}(\tau) = m[v(s(\tau))\,.\,V]^{-1}v^{\mu}(s(\tau))v^{\nu}(s(\tau)). \tag{6}$$

where $s(\tau)$ is the value of the proper time of the particle at the event where the world line of the particle intersects the observer's hyperplane $\Sigma(\tau)$.

In order to prove that $v\,.\,V\,\mathrm{d}s\,\mathrm{d}V(V) = \mathrm{d}^4x$ note that from (2) we have $v\,.\,V\,\mathrm{d}s\,\mathrm{d}V(V) = \mathrm{d}s\,v\,.\,\mathrm{d}\sigma$. On transforming to the reference frame in which the particle is stationary the right-hand side becomes equal to $\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}x'\,\mathrm{d}t' = \mathrm{d}^4x'$, recall that s is the proper time of the particle. But the invariant d^4x' is simply d^4x in the original reference frame, hence the required result.

From (6) and (4) we have for the momentum and energy of the particle

$$\mathscr{P}(\tau) = mv(s(\tau)), \qquad \mathscr{E}(\tau) = mv(s(\tau))\,.\,V.$$

This result is usually written in terms of the proper time of the particle. Invert $s = s(\tau)$ to get $\tau = \tau(s)$, then with $\mathscr{P}(\tau) = \mathscr{P}(\tau(s)) = \mathscr{P}(s)$ the above equations can be written

$$\mathscr{P}(s) = mv(s), \qquad \mathscr{E}(s) = mv(s)\,.\,V$$

which are the usual expressions and give exactly the same numerical values. However, in this latter form it is not clear how to generalise them for many particles.

For n particles T is generalised to

$$T(x) = \sum_{k=1}^{n} m_k \int_{-\infty}^{\infty} \delta(x - z_k(s))v_k(s)v_k(s)\,\mathrm{d}s$$

which leads to

$$\mathscr{P}(\tau) = \sum_{k=1}^{n} m_k v_k(s_k(\tau)), \qquad \mathscr{E}(\tau) = \sum_{k=1}^{n} m_k v_k(s_k(\tau))\,.\,V$$

where $s_k(\tau)$ (the subscript included here for emphasis) is the proper time of the kth particle at which the kth particle's world line intersects the observer's hyperplane $\Sigma(\tau)$. Notice that for a given observer the prescription (3) uniquely defines for each particle the proper time for the term in the summand of $\mathscr{T}(\tau)$. In the usual formulation of special relativistic mechanics of many particles there is an uncertainty in how to specify the proper times of the particle momenta in the sum giving the total momentum, see for example Bergmann (1962, § 20). This difficulty does not arise in my treatment which gives an unambiguous prescription.

From (4), (3) and (2) we obtain

$$\mathscr{P}^{\mu}(\tau) = \int_{\Sigma(\tau)} T^{\mu\nu}\,d\sigma_{\nu}. \tag{7}$$

There are two points concerning (7) which must be emphasised. First, that although very similar to the definition of momentum given in every textbook, namely equation (1), it is not the same. The domain of integration $\Sigma(\tau)$ depends on the observer so that $\mathscr{P}(\tau)$ depends on the observer relative to which it is to be calculated, whereas the choice of the domain of integration in the usual definition depends on the application and is justified by a variety of *ad hoc* arguments. Secondly, that (7) is not the definition of $\mathscr{P}$: $\mathscr{P}$ is defined in terms of $\mathscr{T}$ in (4), and $\mathscr{T}$ is defined in (3). One reason for this is to give the energy-density tensor the central role, the other reason being that the measure of spatial volume $\mathrm{d}V(V)$ has a simple geometrical significance which allows the tensor $\mathcal{T}$ to be defined. By using $\mathrm{d}\sigma(V)$ the relation to spatial volume is obscured and $\mathcal{T}$ cannot be defined.

I shall take (7) as my starting point in the following three sections.

4. Conservation of the total momentum of a free field

Let $\Theta^{\mu\nu}$ be the energy-density tensor for a free electromagnetic field,

$$\Theta^{\mu\nu}=(1/4\pi)(F^{\mu\lambda}F_{\lambda}{}^{\nu}+\tfrac{1}{4}g^{\mu\nu}F^{\iota\lambda}F_{\iota\lambda})$$

where the field F satisfies the Maxwell–Lorentz equations

$$F^{\mu\nu}{}_{,\nu}=4\pi j^{\mu}, \qquad {}^{*}F^{\mu\nu}{}_{,\nu}=0.$$

Then, for a free field with $j^{\mu}=0$,

$$\Theta^{\mu\nu}{}_{,\nu}=0. \tag{8}$$

In order to see how $\mathcal{P}(\tau)$ depends on τ construct the integral

$$\int_{R}\Theta^{\mu\nu}{}_{,\nu}\,\mathrm{d}^{4}x \tag{9}$$

where d^4x is the invariant element of space–time hypervolume expressed in rectilinear coordinates, and R is the region of space–time bounded by the parallel hyperplanes $\Sigma(\tau)$, $\Sigma(\tau')(\tau<\tau')$, and a time-like hypersurface Π at large distance with the surface normal pointing out of R. With the divergence theorem and (8), (9) becomes

$$\int_{\Sigma(\tau)}\Theta^{\mu\nu}\,\mathrm{d}\sigma_{\nu}-\int_{\Sigma(\tau')}\Theta^{\mu\nu}\,\mathrm{d}\sigma_{\nu}+\int_{\Pi}\Theta^{\mu\nu}\,\mathrm{d}\sigma_{\nu}=0. \tag{10}$$

If Θ vanishes sufficiently fast in space-like infinity so that the integral over Π vanishes, then from (7) equation (10) becomes

$$\mathcal{P}^{\mu}(\tau)=\mathcal{P}^{\mu}(\tau')$$

i.e. $\mathcal{P}$ is independent of τ.

This is the same result as given by Rohrlich (1965). However, it should be noted that the orientation of Σ is unambiguously specified in the defining equation (7), whereas in the conventional approach (e.g. Rohrlich 1965, pp 89–91) the orientation of the hyperplane of integration is not built into the definition.

5. Radiation momentum from a particle

Using the radiation part of the energy tensor, $\Theta^{\mu\nu}_{\mathrm{II}}$, defined by Teitelboim (1970) the radiation momentum emitted by a particle between its proper times τ and $\tau+\Delta\tau$ is, from (7),

$$\Delta\mathcal{P}^{\mu}_{\mathrm{II}}(\Delta\Sigma)=\int_{\Delta\Sigma}\Theta^{\mu\nu}_{\mathrm{II}}\,\mathrm{d}\sigma_{\nu} \tag{11}$$

where $\Delta\Sigma$ is that part of Σ (defined in § 2) lying between the future light cones with vertices on the particle world line events at τ and $\tau+\Delta\tau$.

Teitelboim (1970) has proved that the integral in (11) remains unchanged if Σ is translated parallel to itself or tilted or both, i.e. $\Delta\mathscr{P}^{\mu}_{\mathrm{II}}$ is independent of the observer and depends only upon the particle world line, τ and $\Delta\tau$. (For details see Teitelboim (1970), especially the discussion related to his figure 1.) With this result the first part of Teitelboim (1970), § III can be taken over giving

$$\frac{\mathrm{d}\mathscr{P}^{\mu}_{\mathrm{II}}}{\mathrm{d}\tau}=\frac{2e^2}{3c^5}a^{\lambda}a_{\lambda}v^{\mu}.$$

Hence, from (4) we obtain

$$\frac{\mathrm{d}\mathscr{E}_{\mathrm{II}}}{\mathrm{d}\tau}=\frac{2e^2}{3c^5}a^{\lambda}a_{\lambda}v^{\mu}V_{\mu}.$$

6. Velocity-field momentum

The momentum of the velocity field of a particle cannot be defined by simply re-
energy tensor). The reason for this is that the integral diverges on the world line of the
placing $T^{\mu\nu}$ in (7) by $\Theta^{\mu\nu}_{\mathrm{I}}$ ($\equiv\Theta^{\mu\nu}-\Theta^{\mu\nu}_{\mathrm{II}}$, where $\Theta^{\mu\nu}$ is the complete electromagnetic
particle. Let Σ' be the hyperplane Σ (defined in § 2) with a covariantly specified neighbourhood containing the event at which the particle world line intersects Σ removed. Then define, as in (7),

$$\mathscr{P}^{\mu}_{\mathrm{I}}(\tau)=\int_{\Sigma'(\tau)}\Theta^{\mu\nu}_{\mathrm{I}}\,\mathrm{d}\sigma_{\nu}. \tag{12}$$

Before the limit of this neighbourhood shrinking to zero is taken the particle and velocity-field momenta must be renormalised, the mathematical impropriety of this trick should not be overlooked.

Notice that the definition (12), which is a consequence of the general prescription given in § 3, is not in accord with the definitions given and used by Rohrlich (1965) and Teitelboim (1970). These authors define Σ' with its normal orientated parallel to the particle velocity v. While their definition of Σ' certainly simplifies the computation it has a profound disadvantage: in general their definition cannot be generalised for two or more particles, for the orientation of Σ' is then ambiguous. With the definition of § 3 no such difficulty arises because the orientation of Σ' is determined by the observer's velocity V. See my remarks at the end of § 3.

The calculation of the velocity-field momentum is central to the derivation of the Lorentz–Dirac equation. That the definition of $\mathscr{P}^{\mu}_{\mathrm{I}}$ in (12) is different from that used by previous authors suggests that the equation of motion (obtained from momentum balance) will differ from the well known Lorentz–Dirac equation, and it is not hard to see that this is so. Equally, a cursory examination shows that the Schott term will still exist so that the difficulties of runaway solutions and preacceleration will remain, as will the necessity for renormalisation. I hope to publish a paper dealing with these matters in the near future.

7. Conclusion

In this paper I have shown that my definition of an energy tensor, given previously in the context of general relativity, provides a prescription for the treatment of momentum and energy and their conservation theorems in classical electrodynamics. This prescription has been shown to successfully reproduce the well known and desirable theorems of conservation of momentum for free fields and the rate of change of radiation momentum together with their associated results. For a single particle the rate of change of velocity-field momentum is different from that usually given, and this implies a modification of the Lorentz–Dirac equation of motion. These modifications will not eliminate any of the difficulties associated with that equation and its derivation; a proposal to resolve these difficulties will be made in a future publication. Where the usual treatment becomes obscure, in treating the momentum of many particles, my prescription provides an explicit and conceptually simple alternative.

The reader will have remarked that the results concerning momentum and energy could have been obtained by taking equation (7) as the definition of momentum together with (4) for the energy. Comparing (7) with (1) it is seen that this amounts to including the specification of σ (as being Σ) in the definition (1). From this observation it might be supposed that § 2 and § 3 are redundant and could be omitted. In so far as classical electrodynamics is concerned this argument has some force in that the energy-density tensor is in many ways peripheral to the theory. However, in general relativity the energy-density tensor has a central role and forms the natural starting point for a definition of energy and momentum; hence the significance of defining $\mathscr{T}$, and then $\mathscr{P}$ and $\mathscr{E}$ in terms of $\mathscr{T}$. With general relativity in mind it is manifestly desirable to give the electrodynamic energy-density tensor a more central role in classical electrodynamics than it might otherwise be given and to use the same definitions of energy and momentum. It is this which provides the motivation for the form given to this paper.

I have drawn attention to the invariant definition of spatial length and given the corresponding definition of invariant spatial volume. I have also attempted to make clear the relationship of the extension and its norm to the invariant definition of spatial volume. Without this background the true geometrical significance of the integral in (7) and the choice of Σ as the domain of integration is obscured.

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Appendix

In this appendix I give a brief account of how physically significant quantities can be expressed as invariants.

The reference frame, the coordinate system, and the observer are distinct from each other. Thus the walls of a laboratory, the fixed stars, a satellite are examples of frames of reference. In the theory of special relativity the class of inertial reference frames is always used. Given a reference frame a coordinate system can be set up: either rectilinear or any curvilinear system.

An observer is a time-like world line L_0 in space–time and at any instant is characterised by an event X together with a tetrad of mutually orthogonal unit vectors. One of the tetrad is the vector tangent to L_0 at X, namely V. The remaining three form a space-like orthonormal triad in the local rest space of the observer, namely the hyperplane orthogonal to V, Σ. The space–like triad is only defined up to a rotation about the local time axis of the observer, V. For a detailed account of the observer see Sachs and Wu (1977, § 2.1), albeit mathematically rather a sophisticated one. (N.B. My 'observer' is their 'instantaneous observer' with a space-like triad.) In the theory of special relativity the observer is usually taken to be a rectilinear world line and can therefore be put into correspondence with a particular inertial frame.

There are two classes of theoretical entities which are regarded as physically significant, those which are: (i) observer-dependent, or relative; (ii) observer-independent, or intrinsic. For example, the energy of a particle is calculated relative to an assigned observer, whereas the magnitude of the momentum of one particle relative to another is intrinsic to the two particles. Both classes are independent of the coordinate system and reference frame. This independence is ensured in the theory by the use of the tensor calculus, preferably in a coordinate-free notation, and by expressing all physically significant quantities as scalar invariants. Of course a physical phenomenon, such as the motion of a system of particles, does not depend on the reference frame, the coordinate system, or the observer so that one might expect only intrinsic quantities to be physically significant. This is true insofar as the physical system is concerned, but it is convenient in the theory to define some quantities relative to an assigned observer, for example the energy.

I now illustrate these remarks with two examples. First, the energy of a particle relative to the observer (X, V) is $\varepsilon = -p \,.\, V$, where $p = mv$, and both v and V are relative to the same (inertial) reference frame. Since ε is a scalar invariant, neither a change of reference frame nor the choice of coordinates will effect its value; the form of $\varepsilon = -p \,.\, V$ expresses this invariance explicitly. To see that ε is indeed the energy let us consider the situation usually described in the textbooks. The reference frame is the observer's reference frame, then using Galilean coordinates we have $V^1 = V^2 = V^3 = 0$, $V^4 = 1$, and $\varepsilon = -p_4 = p^4 = mv^4$; this is the usual result. Since $\varepsilon = -p \,.\, V$ is independent of the reference frame we can say that the energy depends on the momentum of the particle relative to the observer, and nothing else, see Misner *et al* (1973, p 65) and Sachs and Wu (1977, § 3.1.2).

Secondly, if the momentum of particle a is p_a and of b is p_b then their relative momentum is $p_a - p_b$ and its magnitude is $\|p_a - p_b\|$, which is a scalar invariant and is quite independent of the observer.

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